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## LIGHT-CONE SUPERSYMMETRY AND D-BRANES.

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### ABSTRACT

*D*-brane boundary states for type II superstrings are constructed by enforcing the conditions that preserve half of the space-time supersymmetry. A light-cone coordinate frame is used where time is identified as one of the coordinates transverse to the brane's (euclidean) world-volume so that the *p*-brane is treated as a  $(p + 1)$ -instanton. The boundary states have the superspace interpretation of top or bottom states in a light-cone string superfield. The presence of a non-trivial open-string boundary condensate give rise to the familiar *D*-brane source terms that determine the (linearized) Born–Infeld-like effective actions for *p*-branes and the (linearized) equations of motion for the massless fields implied by the usual *p*-brane ansatz. The ‘energy’ due to closed string exchange between separate *D*-branes is calculated (to lowest order in the string coupling) in situations with pairs of parallel, intersecting as well as orthogonal branes – in which case the unbroken supersymmetry may be reduced. Configurations of more than two branes are also considered in situations in which the supersymmetry is reduced to 1/8 or 1/16 of the full amount. The Ward identities resulting from the non-linearly realized broken space-time supersymmetry in the presence of a *D*-brane are also discussed.

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# 1 Introduction

The description of the  $p$ -branes of the  $R \otimes R$  sector of type II superstring theories in terms of  $D$ -branes [1, 2, 3] has provided a string-theoretic interpretation of the world-volume theories of these objects. According to the  $D$ -brane description the  $R \otimes R$  sector  $p$ -branes are described by configurations of open superstrings with end-points tethered on a  $(p + 1)$ -dimensional hypersurface embedded in ten dimensions. The coordinates of the open-string end-points are restricted to the plane  $X^i = y^i$ , where the directions labelled  $i$  are transverse to the brane world-volume ( $i = p + 1, \dots, 9$ ). The dynamics of the brane is therefore prescribed by the open superstring theory with Neumann boundary conditions in the directions labelled by  $\alpha = 0, \dots, p$  and Dirichlet boundary conditions in the transverse directions. The usual kind of ‘effective’ world-volume field theory emerges as an approximation to this ‘underlying’ open superstring theory. The world-volume actions that describe the various branes [4, 5] are generally non-renormalizable  $(p + 1)$ -dimensional supersymmetric field theories that are generalizations of the Nambu–Goto action [6] and the covariant superstring action to higher dimensions. The fact that such  $D$ -branes are described by *open* superstring theory means that they preserve half the supersymmetry of the fundamental type II string theory. This agrees with the fact that they are BPS solitons of the type II theories.

The effective low-energy action for superstring theory in the presence of a solitonic  $R \otimes R$   $p$ -brane can, in principle, be written in the form

$$S = S_{bulk} + S_{source}^p, \tag{1}$$

where the bulk action is a ten-dimensional integral while the source is restricted to the  $p + 1$ -dimensional world-volume. The equations of motion that follow from this action arise in the underlying string theory from the consistency conditions that ensure conformal invariance of the world-sheet theory (in the case of the type IIB theory there is no obvious covariant bulk action but the source-free equations of motion are known [7, 8]). These bulk terms arise from the usual closed-string sector of the theory but the source terms come from world-sheets with boundaries on which there can be a condensate of the open-string fields. These source actions, which are of the Born–Infeld type, reproduce the equations of motion that follow from the consistency conditions for string theory in the background that includes an open-string boundary condensate,  $F = dA$ , of the electromagnetic field associated with the abelian vector potential that arises as a massless open-string state. Such consistency conditions on open-string theory were originally derived in [9, 10, 11] in the case of the type I theory (which has Neumann boundary conditions in all ten space-time dimensions).

The open-string world-sheet spanned by a ground-state fluctuation of a  $D$ -brane is a disk with the appropriate boundary conditions – Neumann in  $p + 1$  dimensions and Dirichlet in the remaining  $9 - p$ . This world-sheet can be represented by a semi-infinite

cylinder describing the evolution of the vacuum state at  $\tau = -\infty$  to the boundary (which we shall take to be at  $\tau = 0$ ). With a closed-string state in the  $R \otimes R$  or  $NS \otimes NS$  sector coupling to the disk the semi-infinite cylinder describes the evolution of a closed string state  $|\Phi\rangle$  from  $\tau = -\infty$  to the boundary at  $\tau = 0$ ,

$$\langle\Phi\rangle_{Disk} = \langle\Phi|B, \eta\rangle, \quad (2)$$

where  $|B, \eta\rangle$  denotes the boundary state and  $\eta = \pm 1$  labels whether the state is BPS or anti-BPS. In the formalism with world-sheet supersymmetry there are separate boundary states in each sector,  $|B, \eta\rangle_N$  and  $|B, \eta\rangle_R$ , which both satisfy the boundary conditions

$$(\partial X^i - \bar{\partial} X^i)|B, \eta\rangle = 0, \quad (\partial X^\alpha + \bar{\partial} X^\alpha)|B, \eta\rangle = 0, \quad (3)$$

that impose Neumann conditions on  $p+1$  directions and Dirichlet conditions on the rest. In order for the boundary state to preserve the superconformal invariance of the bulk CFT, the super stress-energy tensor has to satisfy continuity conditions at the boundary, which take the form [1, 13]

$$(F + i\eta\tilde{F})|B, \eta\rangle = 0, \quad (4)$$

where the generators of world-sheet supersymmetry transformations are defined by  $F = \psi^\mu \partial X^\mu$  and  $\tilde{F} = \tilde{\psi}^\mu \bar{\partial} X^\mu$ . These conditions preserve half the world-sheet supersymmetry. The boundary conditions on the world-sheet fermions follow from the consistency of (4) with (3),

$$(\psi^i + i\eta\tilde{\psi}^i)|B, \eta\rangle = 0, \quad (\psi^\alpha - i\eta\tilde{\psi}^\alpha)|B, \eta\rangle = 0. \quad (5)$$

The fact that these states preserve half the world-sheet supersymmetry is related to the fact that they also preserve half the space-time supersymmetry which relates the boundary states in the  $R \otimes R$  and  $NS \otimes NS$  sectors,  $|B, \eta\rangle_N$  and  $|B, \eta\rangle_R$  [12].

It is also possible to formulate the problem in a light-cone gauge [13], in which case space-time supersymmetry is manifest. This will be the subject of this paper. In the following discussion the light-cone ‘time’ coordinate will be taken to lie along the axis of the cylinder which is one of the directions *transverse* to the brane. In this parameterization  $X^+ = x^+ + p^+ \tau$  and the momentum component  $P^+$  is constant. Since the boundary state,  $|B, \eta\rangle$ , is at a fixed ‘time’  $X^+$  satisfies a Dirichlet boundary condition. Furthermore,  $X^-$  is determined in terms of the transverse  $X^I$  coordinates ( $I = 1, 2, \dots, 8$ ) and the world-sheet fermionic  $SO(8)$  spinor coordinates  $S^a$  and  $\tilde{S}^a$  ( $a = 1, \dots, 8$ ) and also satisfies a Dirichlet boundary condition as will be seen in section 2. This means that there are at least two Dirichlet directions and that one of these is time-like. This kinematics describes a ‘(p+1)-instanton’ rather than a  $D$ -brane (where time is one of the Neumann directions). It is related to the  $D$ -brane by a double Wick rotation. For convenience the words ‘D-brane’ or ‘ $p$ -brane’ will often be used in the following when it is really this Wick rotated version that is under consideration. Since all cases will have at least two Dirichlet directions the value of  $p$  will be restricted to  $-1 \leq p \leq 7$  in the following.

The generic light-cone boundary state that preserves half of the 32 space-time supersymmetry components of the type II theories will be obtained in section 2. It is a generalization to  $p > -1$  of the point-like state that describes the  $D$ -instanton [14] and can be related to the  $p = -1$  case by a finite  $SO(8)$  rotation. Such a state has an interpretation as the top or bottom state of a light-cone superfield (the two cases corresponding to BPS and anti-BPS states). The general description of a boundary state must take into account the possibility of a non-trivial boundary condensate of the open-string fields, in particular the massless abelian gauge potential. This light-cone boundary state has a form that automatically combines the  $NS \otimes NS$  and  $R \otimes R$  sectors.

A single boundary is interpreted as a linearized source of the closed-string fields that contributes to  $S_{source}$  in the effective action. In section 3 it will be seen that the light-cone description of the source agrees, at linearized order, with the source terms that define the black  $p$ -brane solutions of the  $R \otimes R$  sector [15, 18]. In the presence of a boundary condensate the effective source actions are expected to be of the Born–Infeld form [19, 20, 21, 22]. Expanding these to linear order in the bulk fields also reproduces the light-cone frame boundary sources.

Configurations of two or more branes are considered in section 4. The force between two  $D$ -branes is determined to lowest order by the world-sheet that is a cylinder with one boundary lying in the world-volume of each brane (this generalizes the diagram for the  $D$ -instanton of [13, 14, 23]). The force vanishes between parallel identical branes – this is due to a cancellation between the closed-string exchanges in the  $NS \otimes NS$  and  $R \otimes R$  sectors which is characteristic of a BPS system that preserves 1/2 the total space-time supersymmetry. More generally two (non-identical) branes may be parallel and separated, they may intersect (when they share at least one world-volume direction but are not parallel) or they may be orthogonal (when all the euclidean world-volume directions are orthogonal). All possibilities are considered that are consistent with the choice of light-cone frame (the two world-volumes can occupy up to eight dimensions transverse to the  $x^\pm$  directions). The number of unbroken space-time supersymmetries may be 1/2 or 1/4 of the total as is easily deduced by considering a  $SO(8)$  rotation that reduces any configuration to a standard one. Configurations of two branes that preserve 1/4 of the total supersymmetry again exert no force on each other but in this case the force cancels separately within the  $NS \otimes NS$  and  $R \otimes R$  sectors. Various configurations of type IIA and type IIB theories with three and four branes are also considered in section 4, in which case the minimum unbroken supersymmetry may be  $32/2^n$ , where  $n$  is the number of branes. One example is the intersection of three three-branes with world-volumes that each share two common axes – when compactified on  $T^6$  this describes a black hole with  $a = 0$ . The results of this section are light-cone versions of the description of intersecting branes given in [2].

The spontaneously broken space-time supersymmetries are considered in section 5. These non-linearly realized symmetries relate S-matrix elements with different numbers

of soft fermions.

## 2 Space-time Supersymmetry

In the light-cone parameterization of the type IIB theory in which  $X^+ = x^+ + p^+\tau$  the coordinate  $X^-$  is determined by the relation,

$$p^+ \partial_\sigma X^- = \partial_\tau X^I \partial_\sigma X^I - i S^a \partial_{\tau-\sigma} S^a + i \tilde{S}^a \partial_{\tau+\sigma} \tilde{S}^a \quad (6)$$

(whereas the right-moving spinor in the type IIA theory is a dotted  $SO(8)$  spinor). We can anticipate that the boundary conditions on the fermionic coordinates will reflect a  $S$  into  $iM\tilde{S}$  where  $M$  is an orthogonal matrix so that the fermionic terms will cancel in the expression for  $\partial_\sigma X^-|B, \eta\rangle$ . It follows that  $X^-$  satisfies the Dirichlet condition,

$$\partial_\sigma X^- = 0, \quad (7)$$

whether the  $X^I$  are Dirichlet or Neumann coordinates.

The coordinates transverse to the  $\pm$  directions in (3) satisfy the boundary conditions

$$(\partial X^I - M^{pI}_J \bar{\partial} X^J)|B, \eta\rangle_{(p)} = 0, \quad (8)$$

where  $M^{pI}_J$  is an element of  $SO(8)$ . The Neumann directions will be chosen to be  $\alpha = I = 1, \dots, p+1$  while the Dirichlet directions will be chosen to be  $i = I = p+2, \dots, 8$  (the superscript and subscript  $p$  will often be dropped in the following when there is no ambiguity). In the absence of a boundary condensate of the massless open-string vector potential  $M_{IJ}$  can be written in block diagonal form,

$$M_{IJ} = \begin{pmatrix} -I_{p+1} & 0 \\ 0 & I_{7-p} \end{pmatrix} \quad (9)$$

(where  $I_q$  indicates the  $(q \times q)$ -dimensional unit matrix). In the more general case in which there is a boundary condensate of the open-string vector potential (to be considered in the next section)  $M_{IJ}$  is a more general  $SO(8)$  matrix and can be written as,

$$M_{IJ} = \exp \left\{ \Omega_{KL} \Sigma_{IJ}^{KL} \right\}, \quad (10)$$

where  $\Sigma_{IJ}^{KL} = (\delta_I^K \delta_J^L - \delta_I^L \delta_J^K)$  are generators of  $SO(8)$  transformations in the vector representation. The parameters  $\Omega_{IJ}$  depend on the open-string boundary condensate and in a particular basis they can be written in block off-diagonal form as,

$$\Omega_{\alpha\beta} = \text{diag}(D_1, D_2, \dots, D_{(p-1)/2}), \quad (11)$$

with

$$D_m = \begin{pmatrix} 0 & -d_m \\ d_m & 0 \end{pmatrix}, \quad (12)$$

while the components  $\Omega_{ij}$  and  $\Omega_{\alpha i}$  may be taken to vanish for a static  $p$ -brane [24, 25]. In the absence of a condensate  $d_m = \pi$ . In other words the Neumann boundary conditions in  $p + 1$  directions are represented by a rotation of  $\pm\pi$  transverse to  $(p + 1)/2$  axes. In the presence of a condensate of the open-string vector potential the rotations differ from  $\pi$ .

Each of the sixteen-component supercharges of the type II theories decompose in the light-cone treatment into two inequivalent  $SO(8)$  spinors defined in terms of the world-sheet fields  $X^I$  and  $S^a$  ( $a = 1, \dots, 8$ ) by,

$$Q^a = \frac{1}{\sqrt{2p^+}} \int_0^\pi d\sigma S^a(\sigma), \quad Q^{\dot{a}} = \frac{1}{\pi\sqrt{p^+}} \int_0^\pi d\sigma \gamma_{\dot{a}b}^I \partial X^I S^b(\sigma), \quad (13)$$

for the left-moving charges and similar expressions for the right-moving charges,  $\tilde{Q}^a$  and  $\tilde{Q}^{\dot{a}}$  expressed in terms of the right-moving coordinates. The undotted supercharge acts linearly while the dotted supercharge acts non-linearly on the world-sheet fields (and  $\gamma_{a\dot{a}}^I$  are the usual  $SO(8)$  gamma matrices). These charges realize the  $N = 2$  supersymmetry algebra in the light-cone gauge

$$\{Q^a, Q^b\} = 2p^+ \delta^{ab}, \quad \{Q^{\dot{a}}, Q^{\dot{b}}\} = \delta^{\dot{a}\dot{b}} P^-, \quad \{Q^a, Q^{\dot{a}}\} = \frac{1}{\sqrt{2}} \gamma_{a\dot{a}}^I p^I, \quad (14)$$

with a similar algebra for the rightmoving charges. The closed-string light-cone hamiltonian,  $P_{cl}^-$ , is defined by  $P_{cl}^- = P^- + \tilde{P}^-$ , where

$$P^- = \frac{1}{4p^+} (p^\alpha)^2 + \frac{1}{2p^+} \sum_{n=1}^\infty \left( \alpha_{-n}^I \alpha_n^I + n S_{-n}^a S_n^a \right), \quad (15)$$

and the modes of  $X$ ,  $S$  and  $\tilde{S}$  are defined in the usual manner.

In the type IIB theory both the left-moving and right-moving linearly realized supercharges are undotted spinors whereas in the type IIA theory the dotted and undotted indices are switched between the left-moving and right-moving charges. The case of the type IIB theory will be described first.

The boundary state is defined to conserve the linear combinations of space-time supercharges,

$$\begin{aligned} Q_\eta^{+a} |B, \eta\rangle &\equiv (Q^a + i\eta M_{ab} \tilde{Q}^b) |B, \eta\rangle = 0, \\ Q_\eta^{+\dot{a}} |B, \eta\rangle &\equiv (Q^{\dot{a}} + i\eta M_{\dot{a}\dot{b}} \tilde{Q}^{\dot{b}}) |B, \eta\rangle = 0, \end{aligned} \quad (16)$$

which generalize the expressions in [13] which applied to the special case of the  $D$ -instanton (for which  $M_{ab} = \delta_{ab}$ ,  $M_{\dot{a}\dot{b}} = \delta_{\dot{a}\dot{b}}$ ). The remaining combinations

$$Q_\eta^{-a} \equiv (Q^a - i\eta M_{ab} \tilde{Q}^b), \quad Q_\eta^{-\dot{a}} \equiv (Q^{\dot{a}} - i\eta M_{\dot{a}\dot{b}} \tilde{Q}^{\dot{b}}) \quad (17)$$

are the broken supersymmetries that are associated with goldstinos as will be described later.

In order to determine  $M_{ab}$  and  $M_{\dot{a}\dot{b}}$  we first make the ansatz that the  $SO(8)$  fermionic world-sheet fields,  $S^a$  and  $S^{\dot{a}}$ , satisfy the boundary conditions,

$$(S_n^a + i\eta M_{ab} \tilde{S}_{-n}^b)|B, \eta\rangle = 0, \quad (S_n^{\dot{a}} + i\eta M_{\dot{a}\dot{b}} \tilde{S}_{-n}^{\dot{b}})|B, \eta\rangle = 0. \quad (18)$$

The bispinor matrices  $M_{ab}$  and  $M_{\dot{a}\dot{b}}$  are now determined by consistency with the superalgebra. Thus, multiplying the first equation in (18) by  $(S_{-n}^a + i\eta M_{ab} \tilde{S}_n^b)$  determines that  $M$  is orthogonal,

$$(M^T)_{ab} M_{bc} = \delta_{ac}. \quad (19)$$

Furthermore, multiplying the second equation in (18) by  $(S_{-n}^{\dot{a}} + i\eta M_{\dot{a}\dot{b}} \tilde{S}_n^{\dot{b}})$  gives the condition (using (8) and the definition of the nonlinearly realized supercharges),

$$\gamma_{\dot{a}a}^I M^{IJ} - M_{ab} M_{\dot{a}\dot{b}} \gamma_{\dot{b}b}^J = 0. \quad (20)$$

These two conditions are solved by  $SO(8)$  rotations acting on the spinors,

$$M_{ab} = \exp\{\frac{1}{2}\Omega_{IJ}\gamma_{ab}^{IJ}\}, \quad M_{\dot{a}\dot{b}} = \exp\{\frac{1}{2}\Omega_{IJ}\gamma_{\dot{a}\dot{b}}^{IJ}\}, \quad (21)$$

where  $\Omega_{IJ}$  is the same antisymmetric matrix (11) that defined the  $SO(8)$  rotation in the vector basis and  $\gamma^{IJ} = \frac{1}{2}(\gamma^I\gamma^J - \gamma^J\gamma^I)$ . In the absence of a boundary condensate these conditions reduce to

$$M_{ab} = (\gamma^1\gamma^2 \dots \gamma^{p+1})_{ab}, \quad M_{\dot{a}\dot{b}} = (\gamma^1\gamma^2 \dots \gamma^{p+1})_{\dot{a}\dot{b}}. \quad (22)$$

The three matrices  $M_{IJ}$ ,  $M_{ab}$  and  $M_{\dot{a}\dot{b}}$  are related to each other by triality.

The boundary state that solves (16) can now be obtained explicitly as,

$$\begin{aligned} |B\rangle &= \exp \sum_{n>0} \left( \frac{1}{n} M_{IJ} \alpha_{-n}^I \tilde{\alpha}_{-n}^J - i M_{ab} S_{-n}^a \tilde{S}_{-n}^b \right) |B_0\rangle, \\ &= R(M) \exp \sum_{n>0} \left( \frac{1}{n} \alpha_{-n}^I \tilde{\alpha}_{-n}^I - i S_{-n}^a \tilde{S}_{-n}^a \right) |B_0\rangle, \end{aligned} \quad (23)$$

where the zero-mode factor is

$$|B_0\rangle = C \left( M_{IJ} |I\rangle |J\rangle + i M_{\dot{a}\dot{b}} |\dot{a}\rangle |\dot{b}\rangle \right) \quad (24)$$

(and is annihilated by all the positive modes). The argument  $\eta$  has been dropped since a state with one value of  $\eta$  (an anti-BPS state, say) can be transformed into a state with the opposite value (a BPS state) by a  $2\pi$  rotation about all axes – this leaves  $M_{IJ}$  unchanged but reverses the sign of  $M_{ab}$  and  $M_{\dot{a}\dot{b}}$ . In verifying that this state satisfies (16) use is made of the relations  $S_0^a|I\rangle = \gamma_{a\dot{a}}^I|\dot{a}\rangle/\sqrt{2}$  and  $S_0^a|\dot{a}\rangle = \gamma_{a\dot{a}}^I|I\rangle/\sqrt{2}$ . The normalization constant  $C$  can be determined by constructing a cylindrical world-sheet by joining two boundaries together with a closed-string propagator. The cylinder is equivalent to an annulus that can be uniquely determined as a trace over open-string states with end-points fixed on the respective branes. In the absence of a condensate of the open-string field on the boundaries  $C = 1$  but, more generally, its value depends on the boundary condensate [10, 26].

The analogous boundary state for the type IIA theory involves a matrix  $M_{ab}$  that is the product of an odd number of  $\gamma$  matrices.

The operator  $R(M)$  in (23) is the representation of  $SO(8)$  rotations on the non-zero modes and is defined by

$$R(M) = \exp \sum_{n>0} \left( \frac{1}{n} T_{IJ}^{(\alpha)} \alpha_{-n}^I \alpha_n^J + T_{ab}^{(S)} S_{-n}^a S_n^b \right), \quad (25)$$

where

$$T_{IJ}^{(\alpha)} = \Omega_{KL} \Sigma_{IJ}^{KL}, \quad T_{ab}^{(S)} = \frac{1}{2} \Omega_{KL} \gamma_{ab}^{KL}. \quad (26)$$

This satisfies the group property  $R(M_1)R(M_2) = R(M_1M_2)$ . Similarly, apart from the overall scale  $C$ , the zero-mode part of the state (24) is a rotation of ground-state scalars, which can be written symbolically as

$$|B_0\rangle_{(p)} = CR_0(M^p) (|I\rangle|I\rangle + i|\dot{a}\rangle|\dot{a}\rangle) = CR_0(M^p)|B_0\rangle_{(-1)}. \quad (27)$$

Thus all the  $D$ -brane boundary states are obtained, up to a normalization, by  $SO(8)$  rotations of the boundary state of the  $D$ -instanton (the purely Dirichlet case,  $p = -1$ ),

$$|B\rangle_{(p)} = C\hat{R}(M^p)|B\rangle_{(-1)}, \quad (28)$$

where  $\hat{R} = R_0R$ . The cases with  $p > -1$  are obtained by rotations through  $\pi$  around  $(p+1)/2$  axes while non-zero condensates of the open-string vector potential are determined by continuous rotations. A rotation of  $2\pi$  around all the axes changes a BPS state into an anti-BPS state.

### *Light-cone superfields*

The boundary state can also be expressed as a light-cone superfield by introducing Grassmann coordinates defined by,

$$\theta^a = \frac{1}{2}(Q^a - i\tilde{Q}^a) \quad (29)$$



which is conjugate to

$$\frac{\partial}{\partial \theta^a} = \frac{1}{2p^+}(Q^a + i\tilde{Q}^a). \quad (30)$$

The bosonic sector of the closed-string superfield boundary state can then be expressed as a power series in the  $\theta^a$  by writing the zero-mode factor as,

$$|B_0, \theta, \eta\rangle = \sum_{N=0}^4 \frac{1}{(2N)!} (ip^+)^{N-2} A_{a_1 a_2 \dots a_{2N}}^{2N} \theta^{a_1} \theta^{a_2} \dots \theta^{a_{2N}} |0, \eta\rangle. \quad (31)$$

The coefficients in this expansion are complex functions that satisfy the constraint implied by the ‘reality’ condition,  $A^p = (A^{8-p})^*$  so that there are  $2^8$  real bosonic states.

The supersymmetry conditions (16) restrict the boundary states so that in the case of the  $D$ -instanton ( $p = -1$ ) the BPS and anti-BPS states satisfy the superspace conditions

$$\frac{\partial}{\partial \theta^a} |B, \theta, +\rangle = 0, \quad \theta^a |B, \theta, -\rangle = 0, \quad (32)$$

which means that the boundary states are the top or bottom components of the superfield, (31) depending on whether they are BPS or anti-BPS.

With  $p > -1$  the conditions (16) can be satisfied by requiring linear combinations of  $\theta$  and  $\partial/\partial\theta$  to annihilate the state. This is conveniently expressed in terms of a modified Grassmann coordinate,

$$\hat{\theta} = \frac{1}{2}(1 + M)_{ab} \theta^b + \frac{p^+}{2}(1 - M)_{ab} \frac{\partial}{\partial \theta^b} \quad (33)$$

and its conjugate

$$\frac{\partial}{\partial \hat{\theta}^a} = \frac{1}{2p^+}(1 - M)_{ab} \theta^b + \frac{1}{2}(1 + M)_{ab} \frac{\partial}{\partial \theta^b}. \quad (34)$$

The components of the superfield that is a function of  $\hat{\theta}'$  are linear combinations of the components of the field (31). The conditions (16) are satisfied in general by

$$\frac{\partial}{\partial \hat{\theta}^a} |B_0, \hat{\theta}, +\rangle = 0, \quad \hat{\theta}^a |B_0, \hat{\theta}, -\rangle = 0. \quad (35)$$

This means that the boundary state is the top or bottom component of the light-cone superfield defined as an expansion in  $\hat{\theta}$ . This applies for any value of  $p$ .

The complete boundary state may be expressed in a string superspace by introducing a Grassmann  $SO(8)$  world-sheet spinor coordinate,  $\hat{\Theta}^a$  (with zero mode  $\hat{\theta}^a$ ), and defining a string superfield  $\Phi[X^I, \hat{\Theta}^a, p^+, x^+] = \langle X^I, \Theta^a, p^+, x^+ | \Phi \rangle$ , where  $|\Phi\rangle$  denotes a general closed-string state. The string field theory source term takes the form,

$$\langle \Phi | B \rangle = \int dx^+ dp^+ D^8 X^I D^8 \hat{\Theta}^a \delta^{7-p}(X^i) \delta^8(\hat{\Theta}) \delta(x^+) \Phi \quad (36)$$

( $i = p + 2, \dots, 8$ ) where  $\int D^8 \hat{\Theta} \delta^8(\hat{\Theta})$  picks out the bottom state in the (infinite-dimensional) stringy light-cone supermultiplet, which is the BPS state. The source for the anti-BPS state would not contain the factor  $\delta^8(\hat{\Theta})$ . The full light-cone string field theory action in the presence of the BPS source term has the form,

$$S = \langle \Phi | 2p^+ (P^- - p^-) | \Phi \rangle + \langle \Phi | B \rangle + \text{interaction terms}, \quad (37)$$

where the integration over the super space-time coordinates,  $X^I$ ,  $\hat{\Theta}$ ,  $p^+$  and  $x^+$  is implied by the notation [27].

### 3 Source equations

The zero-mode factor in the boundary state determines the coupling of the  $D$ -brane to the massless fields so the relative strength of the source terms in the effective string action  $S_{\text{source}}$  in (1) can be seen from (24). Decomposing  $M_{IJ}$  into  $SO(8)$  representations  $8 \otimes 8 = 35 + 28 + 1$ ,

$$M_{IJ} = M_{(IJ)} + M_{[IJ]} + \frac{1}{8} \eta_{IJ} M_{KK} \quad (38)$$

where  $(\ )$  denotes the symmetric traceless part and  $[\ ]$  denotes the antisymmetric part. These terms couple to the transverse graviton,  $G_{IJ}$ , and antisymmetric tensor,  $B_{IJ}^N$ , as well as to the dilaton of the type II theories. The  $R \otimes R$  component in the boundary state can be written as the sum of  $SO(8)$  representations,

$$M_{\dot{a}\dot{b}} = \frac{1}{8} \delta_{\dot{a}\dot{b}} \text{tr} M + \frac{1}{16} \gamma_{\dot{a}\dot{b}}^{IJ} \text{tr}(\gamma^{IJ} M) + \frac{1}{384} \gamma_{\dot{a}\dot{b}}^{IJKL} \text{tr}(\gamma^{IJKL} M), \quad (39)$$

which defines the couplings of the boundary state to the transverse components of the massless  $R \otimes R$  potentials,  $\chi$ ,  $B_{IJ}^R$  and  $A_{IJKL}^{(4)}$  (which is self-dual in the transverse space). The information in the boundary state thus determines the relative strengths of the sources for the fields in the theory that modify the bulk field equations.

Covariant string perturbation theory takes place in the linearized Siegel gauge,

$$\partial_\nu h^\nu{}_\mu - \frac{1}{2} \partial_\mu h^\nu{}_\nu + 2 \partial_\mu \phi = 0, \quad (40)$$

where  $h_{\mu\nu} = G_{\mu\nu} - \eta_{\mu\nu}$  is the metric fluctuation. The light-cone gauge is obtained by setting  $h^{+\nu} = 0$ . The light-cone gauge graviton is then identified with the traceless part of the metric fluctuation,  $\hat{h}_{IJ} = h_{IJ} - \eta_{IJ} h^K{}_K / 8$  and the dilaton is proportional to the trace of the metric fluctuation,

$$\hat{\phi} = \phi - \phi_0 = \frac{1}{4} h^I{}_I = \frac{1}{4} (h^i{}_i + h^\alpha{}_\alpha). \quad (41)$$

In the space-time supersymmetric light-cone gauge the fluctuations of all the fields are normalized in a manner that is independent of the coupling constant,  $g = e^{\phi_0}$ . However, in the covariant action the terms in the NS $\otimes$ NS sector have a normalization factor of  $1/g^2$ . This means that in making a comparison between the light-cone gauge results and the expansion of the covariant action in small fluctuations it will be necessary to redefine the NS $\otimes$ NS fields by a factor of  $g$ . In other words, denoting the covariant field fluctuations by a tilde, we shall make the identification  $h_{IJ} = \tilde{h}_{IJ}/g$  for the metric fluctuation,  $\phi = \tilde{\phi}/g$  for the dilaton and  $B_{IJ}^N = \tilde{B}_{IJ}^N/g$  for the antisymmetric tensor.

### 3.1 Black $p$ -brane Ansätze

The values of the boundary sources can be compared directly with the explicit  $p$ -brane solutions [15] of the effective type II supergravity theory. In the absence of a condensate of the open-string vector potential the couplings of the NS $\otimes$ NS fields are determined by the symmetric matrix,  $M_{IJ} = \text{diag}(-I_{p+1}, I_{7-p})$ . In the light-cone frame the graviton  $\hat{h}_{IJ}$  couples to the traceless part of this matrix,

$$\text{diag}\left(-\frac{1}{4}(7-p)I_{p+1}, \frac{1}{4}(p+1)I_{7-p}\right), \quad (42)$$

while the dilaton coupling is proportional to the trace,

$$\frac{1}{4}\text{tr}M_I^I = -\frac{1}{2}(p-3). \quad (43)$$

Similarly, the R $\otimes$ R  $(p+1)$ -form couples to couplings are proportional to  $M_{ab} = (\gamma^1 \dots \gamma^{p+1})_{ab}$ . If the ‘electric’ charge is carried by an object with a  $(p+1)$ -dimensional world-volume the ‘magnetic’ charges are carried by objects with a  $(p'+1)$ -dimensional world-volume where  $M_{ab}^{p'} = *M_{ab}^p$ , where  $*$  denotes the Hodge dual in the eight-dimensional transverse space,  $M^p = \gamma^1 \dots \gamma^{7-p}$ .

The couplings of the  $D$ -brane sources to the dilaton, graviton and R $\otimes$ R potentials determine the source terms in the effective field theory in linearized approximation around flat space. These terms can be compared with those associated with the R $\otimes$ R sector black  $p$ -brane solutions of [15]. These are solutions of the type II supergravity theories that preserve one half of the space-time supersymmetries in which the metric and the dilaton take the following form in the string frame,

$$\begin{aligned} ds^2 &= A^{-1/2}(x^i)dx^\alpha dx^\beta \eta_{\alpha\beta} + A^{1/2}(x^i)dx^i dx^j \eta_{ij} \\ e^\phi &= A^{-(p-3)/4} \end{aligned} \quad (44)$$

Here, the function  $A$  depends only on the transverse coordinates  $x^i$  ( $i = p+2, \dots, 7-p$ ) and satisfies  $\partial^2 A = 0$  for  $x^i \neq 0$ . It is given by,

$$A(x^i) = 1 + \frac{Q}{|x^i|^{7-p}}. \quad (45)$$

where  $Q$  is a (quantized) charge.

To make the comparison with the light-cone boundary state the solution should first be Wick rotated so that the world-volume has euclidean signature and the transverse space has lorentzian signature. The solution (44) can then be expanded in small fluctuations around flat space as  $|x^i| \rightarrow \infty$  (the space-times of black p-branes are asymptotically minkowskian), using

$$A^a = \left(1 + \frac{Q}{|x^i|^{7-p}}\right)^a \sim 1 + a \frac{Q}{|x^i|^{7-p}} + O(|x^i|^{-14+2p}). \quad (46)$$

The second term corresponds to small fluctuations around Minkowski space and is generated by a source of strength proportional to  $a$ . After transforming to the light-cone gauge the metric fluctuations are non-trivial in the transverse space with a trace that is identified with the dilaton while the traceless part determines the physical graviton. The coefficients of these terms agree with those obtained from the massless  $D$ -brane sources. The correspondence between D-branes and the black p-brane solutions can also be analyzed by comparing scattering massless closed string states of D-branes and scattering in the black p-brane background [16, 17].

### 3.2 Boundary condensate

The boundary action that describes the coupling of a condensate of the massless open-string gauge potential has the form

$$\int ds \left( A_\alpha \dot{X}^\alpha - \frac{i}{2} F_{\alpha\beta} S \gamma^{\alpha\beta} S \right). \quad (47)$$

where  $s$  is the parameter on the world-sheet boundary and  $F_{\alpha\beta} = \partial_{[\alpha} A_{\beta]}$ . In addition, if  $B_{\alpha\beta}^N$  is constant the Wess–Zumino term in the bulk action is a total derivative and may be expressed as a surface term so that the total surface term is

$$S_{surface} = \int ds \mathcal{F}_{\alpha\beta} \left( X^{[\alpha} \partial_s X^{\beta]} - i S \gamma^{\alpha\beta} S \right), \quad (48)$$

where  $\mathcal{F} = F - B^N$ . This mixing of the closed-string antisymmetric tensor and the open-string vector was considered in [28]. The quantity in parentheses in this expression is the generator of rotations in the  $\alpha - \beta$  direction. This condensate can be expected to be a consistent string background if the field strength is slowly varying (so that terms involving the derivative of  $\mathcal{F}$  may be dropped), which is all that will be considered here. The presence of this boundary term in the action leads to a modification of the boundary conditions in the Neumann directions, giving  $(\partial_n X^\alpha + \mathcal{F}_{\alpha\beta} \partial_t X^\beta)|B, \eta\rangle = 0$  where  $n$  and  $t$  are the normal and tangential directions to the boundary. These conditions are equivalent to (25) with

$$M_{\alpha\beta}^p = - \left[ (1 - \mathcal{F})(1 + \mathcal{F})^{-1} \right]_{\alpha\beta}. \quad (49)$$

and the normalization constant  $C$  in the end-state  $|B_0\rangle$  may be determined to be,

$$C = \sqrt{\det(1 + \mathcal{F})}. \quad (50)$$

#### NS $\otimes$ NS *sources and Born-Infeld actions*

The boundary source terms lead to modifications of the effective low-energy field theory as was studied in the purely Neumann type I theory in [9, 11]. There, BRST invariance in the presence of boundaries was shown to lead to an effective action that includes a source term,  $S_{source}$ , that has a dependence on the open-string massless vector potential that is of the Born-Infeld form.

In the context of a  $p$ -brane similar arguments again lead to a Born-Infeld-like source action [6]. This is defined in a  $(p+1)$ -dimensional space which has euclidean signature when the kinematics are appropriate to our transverse light-cone gauge. The NS $\otimes$ NS part of this action has the form,

$$S_{NS}^p = \int d^{p+1}x e^{-\phi} \sqrt{\det(G + \mathcal{F})}. \quad (51)$$

This can be compared with the source terms that arise from the massless components in the boundary state (24) by expanding (51) in small fluctuations of the bulk closed-string fields around their constant background values. These fluctuations comprise the dilaton  $\tilde{\phi} = \phi - \phi_0$  and the metric and antisymmetric tensor fields that may be combined into  $\tilde{h}_{\alpha\beta} = \eta_{\alpha\beta} - G_{\alpha\beta} + B_{\alpha\beta}^N$ . Recalling that  $\tilde{\phi} = (\tilde{h}^i_i + \tilde{h}^\alpha_\alpha)/4$  the expansion of the determinant factor in (51) gives

$$\begin{aligned} & -\frac{1}{g} \tilde{\phi} \sqrt{\det(1 + \mathcal{F})} + \frac{1}{2g} \sqrt{\det(1 + \mathcal{F})} \tilde{h}^{\alpha\beta} (1 + \mathcal{F})_{\alpha\beta}^{-1} \\ &= \frac{1}{4g} \sqrt{\det(1 + \mathcal{F})} \left( -\tilde{h}_{ii} + \left( \frac{1 - \mathcal{F}}{1 + \mathcal{F}} \right)_{\alpha\beta} \tilde{h}^{\alpha\beta} \right), \end{aligned} \quad (52)$$

where  $g = e^{\phi_0}$ . Taking into account the fact that the light-cone graviton is defined by  $h_{IJ} = \tilde{h}_{IJ}/g$ , this is precisely the same as the source obtained from the massless part of the boundary state (24).

#### Boundary condensate and R-R fields

The couplings between the boundary field strength  $\mathcal{F}$  and the R  $\otimes$  R sector massless potentials of the type IIB theory are simplest to express in the frame in which  $\mathcal{F}$  is block off-diagonal,

$$\mathcal{F}_{IJ} = \text{diag}(\mathcal{F}_1, \dots, \mathcal{F}_{(p+1)/2}, 0, \dots, 0) \quad (53)$$

where  $\mathcal{F}_i$  is given by

$$\mathcal{F}_i = \begin{pmatrix} 0 & -f_i \\ f_i & 0 \end{pmatrix} \quad (54)$$

(for  $p$  odd). The matrix  $M_{ab}$  which determines the coupling to the  $RR$  states is then given by the following expression

$$M_{ab} = \prod_{i=1}^{(p+1)/2} \frac{1}{(1 + f_i^2)^{1/2}} (1 + f_i \gamma^{2i-1 \ 2i}) \gamma^1 \dots \gamma^{p+1} \quad (55)$$

The contributions to the covariant world-volume source action are determined by this expression. They are of Chern-Simons type,  $\int d^{p+1}x A_R^k (\wedge F)^n$ , where  $A_R^k$  is a  $k$ -form potential in the  $R \otimes R$  sector (and  $k + 2n = p + 1$ ). For example, the matrices  $M_{ab}^p$  for the cases  $p = 1$  and  $p = 3$  are given by

$$M_{ab}^1 = \frac{1}{(1 + f_1^2)^{1/2}} (\gamma_{ab}^{12} + f_1 \delta_{ab}) \quad (56)$$

$$M_{ab}^3 = \frac{1}{(1 + f_1^2)^{1/2} (1 + f_2^2)^{1/2}} (\gamma_{ab}^{1234} + f_2 \gamma_{ab}^{12} + f_1 \gamma_{ab}^{34} + f_1 f_2 \delta_{ab}). \quad (57)$$

Correspondingly, covariant effective euclidean source actions for these fields are given by

$$S_R^1 = i \int d^2x (B^R + \chi \mathcal{F}) \quad (58)$$

$$S_R^3 = i \int d^4x (\frac{1}{2} \chi \mathcal{F} \wedge \mathcal{F} + B^R \wedge \mathcal{F} + A_R^4). \quad (59)$$

Obvious generalizations of these expressions hold for all other values of  $p \leq 7$ , including the even values of relevance to the type IIA theory.

Expanding these actions to linearized order in the bulk fields gives sources for the  $R \otimes R$  fields that correspond to the terms in (56), (57). These terms also contain couplings between the bulk fields, such as  $\chi \wedge B^N$  in (58) and  $B^N \wedge B^N$ ,  $B^R \wedge B^N$  in (59). The presence of such couplings can be demonstrated directly by calculating the disk amplitude with the appropriate insertion of closed-string vertex operators.

### 3.3 Examples

#### *D-instanton*

In the case  $p = -1$  all coordinates satisfy Dirichlet boundary conditions and  $M_{IJ} = \delta_{IJ}$ ,  $M_{ab} = \delta_{ab}$ . The massless terms in the boundary state are

$$|B_0\rangle = \delta_{IJ} |I\rangle |\tilde{J}\rangle + i \delta_{a\dot{b}} |\dot{a}\rangle |\tilde{\dot{b}}\rangle. \quad (60)$$

Thus, the couplings of the source to the dilaton  $\phi$  and the R-R scalar,  $\chi$ , are equal which agrees with the linearized approximation to the  $D$ -instanton ansatz in [29]. The factor of

$i$  indicates that this should be interpreted as a euclidean solution since the pseudoscalar field  $\chi$  is pure imaginary after a Wick rotation to euclidean space.

The full theory includes a sum over diagrams with arbitrary numbers of world-sheet boundaries. The leading term in the partition function comes from the exponential of the disk diagram so that

$$Z = e^{-\frac{1}{g} - i\chi_0}. \quad (61)$$

Connected diagrams with a larger number of boundaries vanish by supersymmetry. The exponent in  $Z$  is the constant part of the euclidean source action,

$$S_{source}^{-1} = \int d^{10}x (e^{-\phi} + i\chi) \delta^{10}(x - x_0). \quad (62)$$

The fact that the  $D$ -instanton has no world-volume dimensions means that there is no dependence on the metric in the source action and hence no coupling to the graviton – it couples to the dilaton and  $\chi$  only. The source term is therefore unchanged in transforming between the string and the Einstein frames.

#### *The $D$ -string*

In the absence of a boundary condensate of the open-string vector potential the  $p = 1$  boundary state couples to the dilaton, the graviton, and the components of the  $R \otimes R$  -antisymmetric tensor  $B_{12}^R$  in the world-sheet directions.

In the presence of a boundary condensate of the open-string vector potential the  $D$ -string also couples to the  $R \otimes R$  scalar,  $\chi$ , and the  $NS \otimes NS$  antisymmetric tensor,  $B^N$  [3]. This is seen in the present light-cone formalism by constructing the boundary state (49) for the case  $p = 1$ , in which the rotation matrices have the form (which is analogous to the covariant description in [12])

$$M_{ab}^1 = \frac{1}{\sqrt{1+f^2}} (\delta_{ab} + f\gamma_{ab}^{12}), \quad (63)$$

and

$$M_{IJ}^1 = \begin{pmatrix} M_{\alpha\beta}^1(f) & 0 \\ 0 & I_6 \end{pmatrix}, \quad (64)$$

where the  $2 \times 2$  matrix  $M_{\alpha\beta}^1(f)$  is given by

$$M_{\alpha\beta}^1 = -\frac{1-\mathcal{F}}{1+\mathcal{F}} = \frac{-1}{1+f^2} \begin{pmatrix} 1-f^2 & -2f \\ 2f & 1-f^2 \end{pmatrix}. \quad (65)$$

where  $f = -\frac{1}{2}\epsilon^{\alpha\beta}\mathcal{F}_{\alpha\beta}$ .

The  $SO(8)$  rotation is through an angle  $\alpha$  in the  $0-1$  plane given by,

$$\cos \alpha = -\frac{1-f^2}{1+f^2}, \quad (66)$$

so that the generators of the rotations in (26) are

$$T_{\alpha\beta}^{(\alpha)} = \alpha \Sigma_{\alpha\beta}^{12}, \quad T_{ab}^{(S)} = \frac{\alpha}{2} \gamma_{ab}^{12}. \quad (67)$$

The zero mode part of the boundary state is given by

$$|B_0\rangle_{(1)} = \sqrt{1 + f^2} M_{IJ} |I\rangle |\tilde{J}\rangle + i(f\delta_{\dot{a}\dot{b}} + \gamma_{\dot{a}\dot{b}}^{12}) |\dot{a}\rangle |\tilde{\dot{b}}\rangle \quad (68)$$

Following the general analysis of section 3.2 the effective world-volume action of the D-string is given by a Born-Infeld-like euclidean action

$$S_{source}^1 = \int d^2x (e^{-\phi} \sqrt{\det(G + \mathcal{F})} + i\mathcal{F}\chi + iB_{12}^R), \quad (69)$$

which reproduces the boundary sources in linearized approximation, as before.

In order to make contact with the dyonic string solutions of [30] the action may be expressed in terms of the integer monopole number of the integrated field strength. This may be accomplished by integrating over the vector potentials in a particular topological sector in which  $\int \mathcal{F} = 2\pi m$  [19, 21, 22]. The result is an action of the Nambu–Goto form,

$$\tilde{S}_{source}^1(m) = \int d^2x \left( \left( e^{-2\phi} + (m + \chi)^2 \right)^2 \sqrt{\det G} + imB^N + iB^R \right), \quad (70)$$

which describes a  $D$ -string with tension  $((m + \chi)^2 + 1/g^2)^{1/2}$ .

### *Other branes*

The case  $p = 3$  is described in the absence of a condensate of the open-string potential by  $M_{IJ}^3 = (-I_4, I_4)$  and  $M_{ab}^3 = i(\gamma^1\gamma^2\gamma^3\gamma^4)_{ab}$ , which is self-dual in the eight-dimensional transverse space. In the presence of a non-zero condensate the world-volume theory for the three-brane is given by 4-dimensional  $N = 4$  supersymmetric Yang-Mills theory. The theory is invariant under electric-magnetic duality transformations when these are accompanied by  $SL(2, Z)$  transformations acting on the closed-string bulk fields [22, 31].

The cases with  $p = 5$  and  $p = 7$  are the magnetic duals of the  $D$ -string and the  $D$ -instanton, respectively. The rotations that define the boundary states are obtained from the  $p = 1$  and  $p = -1$  cases simply by multiplying the matrix  $M_{ab}$  by  $\gamma^1 \cdots \gamma^8$ , which implements Hodge duality in the transverse eight-dimensional transverse space.

In the case of the branes of the type IIA theory the possible values of  $p$  are even and the left-moving and right-moving supercharges are described by  $SO(8)$  spinors of opposite type. Thus, the eight linearly realized supersymmetries that annihilate the boundary state are  $(Q^a + M_{a\dot{a}}\tilde{Q}^{\dot{a}})$ , while the non-linearly realized ones are  $(\tilde{Q}^{\dot{a}} + M_{\dot{a}a}Q^a)$ . The analysis is very similar to the Type IIB cases.



## 4 Configurations with two or more branes

In the sector with two separated  $p$ -branes the leading contribution to the free energy comes from the independent energies defined by a functional integral over two disk world-sheets – one with Dirichlet conditions at  $y_1^i$  and the other at  $y_2^i$ . The energy (per unit volume) between these separated branes is determined to lowest order in the string coupling constant by quantum fluctuations that describe the exchange of a closed string between them. This is represented by a world-sheet diagram with the topology of a cylinder with boundaries that describe the world-lines of the end-points of open strings moving in the world-volume of each brane. The dependence of this energy on the separation of the branes,  $L = |y_2 - y_1|$  determines the force between them. If the branes are viewed as instantons the diagram represents the action per unit euclidean volume.

The general expression for the energy (or action) is given by

$$Z(F_1, F_2, L) = C_1 C_2 \int_0^\infty \frac{dt}{2p^+} {}_{(-1)}\langle B, y_2 | e^{-(P_{cl}^- - p^-)t} \hat{R}(M^{p_2 T} M^{p_1}) | B, y_1 \rangle_{(-1)}, \quad (71)$$

where  $p^- = i\partial/\partial y^+$  and the value of  $\eta$  is taken to be the same for both end-states (both of them are either BPS or anti-BPS). With an appropriate choice of the rotations,  $M^{p_1}$  and  $M^{p_2}$ , and normalizations,  $C_1$  and  $C_2$  (71) represents the energy between branes of arbitrary  $p_1$  and  $p_2$ , oriented in any relative direction and with arbitrary condensates,  $F_1$  and  $F_2$ , of the open-string gauge potentials on each brane (which are assumed to vary only slowly). Furthermore, the force between an anti-BPS state and a BPS state is described by rotating through  $2\pi$  around all axes, thereby changing the sign of  $\eta$  in one of the states. The normalizations  $C_1(F_1)$  and  $C_2(F_2)$  may be determined as before by considering the process to be a trace over the open-string states that propagate around the annulus with their end-points fixed in the respective world-volumes of the branes. Writing  $Z$  as a product of zero-mode and non zero-mode factors,

$$Z(F_1, F_2, L) \equiv Z_0(F_1, F_2) Z_{osc}(M^{p_2 T}(2) M^{p_1}(1)), \quad (72)$$

it is clear that  $Z_0(F_1, F_2, L)$  depends explicitly on  $F_1$  and  $F_2$  separately while  $Z_{osc}(M^{p_2 T}(1) M^{p_1}(1))$  depends only on the relative rotation.

### 4.1 Parallel branes

The fact that D-branes are BPS configurations means that two parallel branes of the same type (the same values of  $\eta$  and of the open-string condensate fields,  $f_\alpha$  on each brane) do not exert a force on each other. The force vanishes identically as follows from the fact that exactly half the space time supersymmetry is broken by this configuration, which leads to the vanishing of this amplitude and all higher contributions with the topology of

a sphere with  $n$  holes cut out. This was observed in the case in which all the boundaries are Dirichlet in [13], in which case the ‘energy’ (actually, the action per unit volume) is

$$Z(0, 0, L) = \int_0^\infty \frac{dt}{2p^+} {}_{(-1)}\langle B, y_2 | e^{-(P_{cl}^- - p^-)t} | B, y_1 \rangle_{(-1)}, \quad (73)$$

which vanishes because of the cancellation between the exchange of massless states in the  $\text{NS} \otimes \text{NS}$  and the  $\text{R} \otimes \text{R}$  sectors,

$$\langle \tilde{I} | \langle I | J \rangle | \tilde{J} \rangle - \langle \tilde{a} | \langle \dot{a} | \dot{b} \rangle | \tilde{b} \rangle = 8 - 8 = 0. \quad (74)$$

For widely separated  $D$ -instantons this vanishing is attributed to a cancellation between the exchanged massless scalar and pseudoscalar states [14]. This generalizes to the case of parallel  $D$ -branes of the same type with arbitrary  $p$  and with the same boundary condensates [1] simply by inserting  $1 = R^\dagger(M^p)R(M^p)$  in the matrix element and using the expression (28). The normalization factor  $C_1 C_2$  is finite so that the energy vanishes in these cases also. The tension of the brane can be deduced by isolating the  $\text{NS} \otimes \text{NS}$  part of (74).

However, the interaction energy between two branes of opposite type (one being BPS and the other anti-BPS) does not vanish. Since a BPS state is converted into an anti-BPS state by a  $2\pi$  rotation around all axes,  $|B, -\rangle = \hat{R}(M(2\pi))|B, +\rangle$ , the cylinder diagram that describes the interaction of a BPS  $p$ -brane and an anti-BPS  $p$ -brane is given by,

$$\begin{aligned} Z_{+-}(0, 0, L) &= \int_0^\infty dt \langle B, y_2, + | e^{-2p^+(P^- - p^-)t} \hat{R}(M(2\pi)) | B, y_1, + \rangle \\ &= \int_0^\infty dt (\pi^2 t)^{(p-9)/2} e^{-(y_1 - y_2)^2 / 4\pi t} \frac{\prod_{n>0} (1 + q^{2n})^8}{\prod_{n>0} (1 - q^{2n})^8}. \end{aligned} \quad (75)$$

where  $q = e^{-\pi t}$  (a special case of this with purely Dirichlet boundary conditions,  $p = -1$ , was considered in [13]).

The cylinder diagram is transformed to an annulus by a modular transformation, in which case the expression for  $Z$  is interpreted in terms of a trace over the states of open strings with end-points fixed in the world-volumes of the two branes. This is the appropriate coordinate frame for describing the singularities of this process as a function of the separation,  $L = |y_2 - y_1|$ . After the standard change of variables,  $t' = 1/t$  and  $w = e^{-\pi t'}$ , (75) becomes,

$$Z_{+-}(L) = \frac{\pi^{(p-9)}}{16} \int \frac{dt'}{t'} t'^{-(p+1)/2} \exp\left(-t' \frac{1}{4\pi} (y_1 - y_2)^2 + \pi t'\right) \frac{\prod (1 - w^{2n-1})^8}{\prod (1 - w^{2n})^8}. \quad (76)$$

The expression possesses singularities at  $L^2 = 4\pi^2(1 - n)$ . The  $n = 0$  term gives a pole at space-like separation while the  $n = 1$  term is a singularity at null separation and for

$n > 1$  the singularities are at time-like separations. This is the  $D$ -brane generalization of observations concerning the analytic structure of the purely Dirichlet amplitudes relevant to the  $D$ -instanton ([13] and references therein). For the  $p$ -brane interpretation  $(y_2 - y_1)$  is necessarily space-like and only the  $n = 0$  term leads to a physical singularity at  $L^2 = 4\pi^2$  [32]. This represents the onset of a tachyonic instability in the system. In this situation it requires more detailed dynamical understanding to determine whether there might be a bound state.

In the light-cone frame appropriate to the  $(p + 1)$ -instanton interpretation  $(y_2 - y_1)^2$  has indefinite sign and the process has singularities on and inside the light-cone as well as a single singularity outside the light-cone.

## 4.2 Parallel $D$ -strings with boundary condensates

When the condensates  $F_1$  and  $F_2$  are different on two otherwise identical parallel branes supersymmetry is broken because the combination of supercharges that annihilates one boundary (16) is different from the combination that annihilates the other. This leads to an interaction energy between the branes. The large-distance behaviour of this energy is dominated by the  $\tau \rightarrow \infty$  limit which picks out the massless modes of the closed-string exchanged between the branes.

For example, in the case of two parallel  $D$ -strings ( $p = 1$ ) the coefficients in (71) are given by  $C_1 = \sqrt{1 + f_1^2}$  and  $C_2 = \sqrt{1 + f_2^2}$ . The zero mode factor in  $Z$  is given by,

$$\begin{aligned} Z_0(F_1, F_2, L) &= \langle B_0, F_1, \eta | B_0, F_2, \eta \rangle \\ &= C_1 C_2 (M_{IJ}^T(2) M_{JI}(1) - M_{ab}(2) M_{ba}(1)) \\ &= \frac{8 + 4(f_1^2 + f_2^2) + 8f_1^2 f_2^2 + 8f_1 f_2}{\sqrt{(1 + f_1^2)(1 + f_2^2)}} - 8(f_1 f_2 + 1). \end{aligned} \quad (77)$$

In the case in which  $f_1 = f_2$  this vanishes as the energy between two identical BPS-saturated branes should. More generally, the expression (77) has the property that for small  $f_1$  and  $f_2$  it behaves as

$$Z_0(f_1, f_2, L) \sim f_1^4 + f_2^4 + 6f_1^2 f_2^2 - 4f_1^3 f_2 - 4f_2^3 f_1 + O(f^5). \quad (78)$$

The lower order terms vanish which corresponds to the fact that the moduli space of parallel  $D$ -strings is flat [24].

The oscillator factor,  $Z_{osc}$ , in (72) can be evaluated exactly using (25). Since the right-moving degrees of freedom do not appear in the rotation operator the expression reduces to a trace over the left-moving oscillator states. The generators (26) can be diagonalized by a unitary transformation,

$$\begin{aligned} \alpha_n^I &\rightarrow U_{IJ} \alpha_n^J & , & \quad \alpha_{-n}^I \rightarrow U_{IJ}^\dagger \alpha_{-n}^J \\ S_n^a &\rightarrow V_{ab} S_n^b & , & \quad S_{-n}^a \rightarrow V_{ab}^\dagger S_{-n}^b. \end{aligned} \quad (79)$$

where  $U$  and  $V$  are defined so that

$$\begin{aligned} U^\dagger T^{(\alpha)} U &= \begin{pmatrix} i\alpha & 0 & 0 \\ 0 & -i\alpha & 0 \\ 0 & 0 & 0_6 \end{pmatrix} \\ V^\dagger T^{(S)} V &= \begin{pmatrix} -i\frac{1}{2}\alpha I_4 & 0 \\ 0 & i\frac{1}{2}\alpha I_4 \end{pmatrix}, \end{aligned} \quad (80)$$

where  $\alpha$  (defined in (66)) is the angle of the relative rotation  $M_1^T M_2$ . The traces over the oscillator states are standard and the result is given by

$$Z_{osc}(\alpha, L) = \pi^4 \int_0^\infty dt t^{-4} \frac{\prod_{n=1}^\infty (1 - q^{2n} e^{i\alpha/2})^4 (1 - q^{2n} e^{-i\alpha/2})^4}{\prod_{n=1}^\infty (1 - q^{2n})^6 (1 - q^{2n} e^{i\alpha}) (1 - q^{2n} e^{-i\alpha})} \exp\left(-\frac{L^2}{4\pi t}\right), \quad (81)$$

where  $q = e^{-\pi t}$  and the zero-mode factor follows from (77). This can be written in a compact form in terms of Jacobi  $\theta$  functions and the Dedekind  $\eta$  function,

$$Z_{osc}(\alpha, L) = \frac{\pi^4 \sin(\alpha/2)}{16 \sin^4(\alpha/4)} \int dt t^{-4} \frac{\theta_1^4(\frac{\alpha}{4\pi} | it)}{\eta^9(it) \theta_1(\frac{\alpha}{2\pi} | it)} \exp\left(-\frac{L^2}{4\pi t}\right), \quad (82)$$

This expression exhibits interesting analytical structure as a function of the separation which is exposed in the short cylinder limit,  $t \rightarrow 0$ . The Jacobi transformation,  $t \rightarrow t' = 1/t$ , transforms the cylinder into an annulus, giving,

$$Z_{osc}(\alpha, L) = \frac{\pi^4 \sin(\alpha/2)}{16 \sin^4(\alpha/4)} \int \frac{dt'}{t'} \frac{\theta_1^4(-it' \frac{\alpha}{4\pi} | it')}{\eta^9(it') \theta_1(-it' \frac{\alpha}{2\pi} | it')} \exp(-t' \frac{L^2}{4\pi}). \quad (83)$$

The complete expression for  $Z$  (72) is given by the product of (77) and (83). The argument of the  $\theta_1$  is imaginary so that in the short cylinder limit,  $t' \rightarrow \infty$ , the behaviour of the integrand of (83) is given by

$$\begin{aligned} \lim_{t' \rightarrow \infty} \frac{e^{-t' L^2/4\pi} \theta_1^4(-it' \frac{\alpha}{4\pi} | it')}{\eta^9(it') \theta_1(-it' \frac{\alpha}{2\pi} | it')} &\sim \lim_{t' \rightarrow \infty} \frac{e^{-t' L^2/4\pi} \sin^4(-i\alpha t'/4)}{\sin(-i\alpha/2t')} \\ &\sim \lim_{t' \rightarrow \infty} e^{-t'(L^2 - 2\pi|\alpha|)/4\pi} + O(e^{-\pi t'}). \end{aligned} \quad (84)$$

Hence, the expression diverges for separations  $L^2 = (y_2 - y_1)^2 \leq 2\pi |\alpha|$ . If the transverse space has Minkowski signature this indicates the presence of a pole in  $L^2$  outside the light-cone as well as an infinite set of poles at values

$$L^2 = 2\pi(|\alpha| - 2\pi n), \quad n = 0, 1, \dots \quad (85)$$

Note that (66) implies that the limit of infinite field strength corresponds to  $\alpha = \pi$ .

After continuing to Minkowski signature the magnetic boundary condensate turns into an electric boundary condensate and the rotation matrices become boosts. This change of signature can be accomplished by  $f \rightarrow if$  which implies  $\alpha \rightarrow i\alpha$ . The argument of the  $\theta$  functions are then real and the integrand of (83) exhibits an infinite number of simple poles on the real  $t'$  axis at  $t' = 2\pi(2k+1)/\alpha$  where  $k$  is integer. This leads to an imaginary part in the energy, which is given by the sum over the residues of the poles. This is very similar to the situation in [33], where the type I superstring was considered, which has Neumann boundary conditions in all directions. In fact, the rather complicated sum over spin structures obtained in [33] reduces to a simple expression of the form (83).

### 4.3 Parallel, intersecting and orthogonal branes.

The most general configuration of two branes is one in which the branes may have different values of  $p$ , they may both be BPS states or one of them may be anti-BPS and they may or may not be parallel. We will be interested in describing which configurations have residual supersymmetries. The important issue of determining whether there are BPS bound states of branes lying within one another will not be addressed in the following. Furthermore, we will not consider the rich class of configurations in which there is a non-trivial boundary condensate.

A systematic way of analyzing configurations is to first consider one of the branes to be a  $D$ -instanton ( $p_1 = -1$ ) and the other one to have  $p_2 = -1, 1, 3, 5$  or  $7$  (the type IIA case is analogous but  $p_1$  and  $p_2$  take even values). All other cases can then be obtained by appropriate rotations.

#### 4.3.1 A $D$ -instanton and a $p$ -brane.

The zero-mode contribution to the cylinder diagram with  $p_1 = -1$  with general  $p_2$  is

$$\begin{aligned} Z_0(p_1 = -1, p_2) &= {}_{(p_2)}\langle B_0 | B_0 \rangle_{(-1)} = {}_{(-1)}\langle B_0 | R_0^{p_2 T} | B_0 \rangle_{(-1)} \\ &= \text{Tr} M_{IJ}^{p_2} - \text{Tr} M_{ab}^{p_2}. \end{aligned} \quad (86)$$

The unbroken supersymmetry generators that annihilate the two boundary states are given by

$$Q_{(-1)}^{+a} = Q^a + i\tilde{Q}^a, \quad Q_{(p_2)}^{+a} = Q^a + iM_{ab}^{p_2}\tilde{Q}^b, \quad (87)$$

(with similar combinations of the dotted supercharges) where the cases of  $\eta = \pm 1$  are included by allowing for the fact that the overall sign of  $M_{ab}^{p_2}$  can be changed by a  $2\pi$  rotation in all directions. The compatibility of these two supersymmetries depends on the value of  $p_2$ . The cases  $p_2$  and  $p'_2 = 6 - p_2$  have similar behaviour.

Thus,  $Z_0 = 0$  when  $p_2 = -1$  or  $p_2 = 7$  (the rotation matrix  $\gamma^1 \cdots \gamma^8 \equiv 1$ ) due to the cancellation between the NS  $\otimes$  NS and R  $\otimes$  R sector as described earlier. In this case both supercharges are identical and there is a total of 16 unbroken supersymmetries.

For  $p_2 = 3$  the two traces in (86) vanish *individually* so that once again  $Z_0 = 0$ , but this time due to the separate vanishing of the  $\text{NS} \otimes \text{NS}$  and  $\text{R} \otimes \text{R}$  sectors. The two matrices  $M_{ab}^{p_r}(r)$  (where  $r = 1, 2$  labels the two branes) are of the form

$$M_{ab}^{-1}(1) = \delta_{ab}, \quad M_{ab}^3(2) = \gamma_{ab}^{1234}. \quad (88)$$

Since  $(\gamma^{1234})^2 = 1$ ,  $M_{ab}^3$  has eigenvalues  $= +1, -1$ . The projector of brane 2 on the eigenvectors with eigenvalue  $+1$  is given by

$$P_{ab}(2) = \frac{1}{2}(1 - \gamma^{1234})_{ab}. \quad (89)$$

The dimensionality of this eigenspace is given by the trace of the projector  $\text{tr}P(2) = 4$ . Therefore, there are four common eigenvalues of  $M_{ab}^{(-1)}(1)$  and  $M_{ab}^3(2)$  and so four undotted supersymmetries remain unbroken. The same reasoning also applies to the dotted supersymmetries. Hence the total number of unbroken supersymmetries is eight – the configuration preserves a quarter of the original supersymmetry. Furthermore, this configuration is invariant under rotation by  $2\pi$  that reverses the overall sign of  $M_{ab}^3$ , which interchanges the BPS and anti-BPS conditions.

When  $p_2 = 1$  or  $p_2 = 5$  the bi-spinor trace vanishes but the bi-vector trace is non-zero so that  $Z_0 \neq 0$ . In this case  $M_{ab}^1 = (\gamma^1 \gamma^2)_{ab}$  has imaginary eigenvalues so that the conserved supercharges of the two branes have no eigenvalues in common – there are no unbroken space-time supersymmetries in this case. This means that there is a force between the branes so that they might form a bound state that can be a BPS state [2, 34].

This behaviour of the zero modes extends to the non-zero mode factor,  $Z_{osc}$ . The non-zero mode contributions from fermionic and bosonic oscillators cancel in the cases  $p_2 = -1, 3, 7$ , so that  $Z_{osc} = 1$ . In the cases  $p_2 = 1, 5$   $Z_{osc}$  is a non-trivial ratio of contributions from bosonic and fermionic oscillators.

#### 4.3.2 Parallel, intersecting and orthogonal $p_1$ and $p_2$ branes.

The cases with  $p_1 > -1$  can be obtained from the above by inserting  $1 = R^{(p_1)T} R^{(p_1)}$  in (86), where  $R^{(p_1)}$  is a rotation that acts on the bi-vector and bi-spinor indices so that,

$$M_{IJ}^{p_1} R_{JK}^{(p_1)} = M_{IK}^{-1}, \quad M_{ab}^{p_1} R_{bc}^{(p_1)} = M_{ac}^{-1}. \quad (90)$$

This is a  $T$ -duality transformation that changes the number of directions in which there are Neumann and Dirichlet boundary conditions. In this way the general case can be reduced to a consideration of the ones considered in (a) above with  $M^{p_2}$  replaced by  $M'^{p_2} = R^{(p_1)} M^{p_2}$ .

The relative orientation of the two branes depends on the choice of  $R^{(p_1)}$ . There are several distinct classes to consider.

- (a) *Parallel branes.* The  $(p_2 + 1)$  world-volume coordinates are a subset of the  $(p_1 + 1)$  world-volume coordinates (where  $p_2 < p_1$ ).
- (b) *Intersecting branes.* A subset of the coordinates of one brane are orthogonal to a subset of the world-volume coordinates of the other.
- (c) *Orthogonal branes.* The world-volume coordinates of one brane are all orthogonal to those of the other. This is not possible for the usual  $p$ -branes since they always share the time direction – but, it is possible for the  $(p + 1)$ -instantons considered in this paper, in which the branes have euclidean world-volumes.

As a specific example, consider the case of a  $D$ -string ( $p_2 = 1$ ) and a three-brane ( $p_1 = 3$ ). Choosing the three-brane world-volume to be in the directions  $I = 1, 2, 3, 4$ , the three distinct configurations of the  $D$ -string correspond to the choices for the  $p_2$  matrices,

$$M_{IJ}^{p_2} = \text{diag}(-1, -1, 1, 1, 1, 1, 1, 1), \quad M_{ab}^{p_2} = \gamma_{ab}^{12}, \quad (91)$$

$$M_{IJ}^{p_2} = \text{diag}(-1, 1, 1, 1, -1, 1, 1, 1), \quad M_{ab}^{p_2} = \gamma_{ab}^{15}, \quad (92)$$

$$M_{IJ}^{p_2} = \text{diag}(1, 1, 1, 1, -1, -1, 1, 1), \quad M_{ab}^{p_2} = \gamma_{ab}^{56}, \quad (93)$$

which describe the parallel, intersecting and orthogonal cases, respectively.

These may now be transformed into the case in which  $p_1 = -1$  by inserting the rotation  $R^{(p_1)}$  that transforms the three-brane into the  $D$ -instanton,

$$R_{IJ}^{(p_1)} = \text{diag}(-1, -1, -1, -1, 1, 1, 1, 1), \quad R_{ab}^{(p_1)} = \gamma_{ab}^{1234}. \quad (94)$$

Acting on the parallel  $D$ -string (91) the result is another  $D$ -string (the matrices  $M^{p_2}$  transform into those of the  $p = 1$  case). Acting on the orthogonal  $D$ -string (93) the result is a  $p_2 = 5$  state. Both of these cases therefore break all the space-time supersymmetries. However, the result of acting with  $R^{(p_1)}$  on the intersecting  $D$ -string (92) is a  $p_2 = 3$  state and so one quarter of the supersymmetries (i.e., 8) are preserved. This is in accord with the results in [35, 2].

Another illustrative example is the case with  $p_1 = 1$  and  $p_2 = 5$  where there are three possible distinct configurations of the string and the fivebrane. In the intersecting case, in which there is one common world-volume direction, no supersymmetries are conserved. The parallel configuration in which there are two common world-volume directions preserves one quarter (i.e., 8) of the supersymmetries. The case in which the world-volumes are orthogonal preserves half (i.e., 16) the space-time supersymmetries, note that the last configuration has no analogue for the  $p$ -branes which share the world-volume time direction.

The general result coincides with that of Polchinski's argument [2]. There, it was shown that the number of directions in which the boundary conditions at either end of

the string are different must be 0 mod 4 if any space-time supersymmetry is preserved. This coincides with the number of  $-1$  entries in the diagonal matrix  $M_{IJ}^{p_1} M_{JK}^{p_2}$ .

The following table summarizes the number of surviving dotted and undotted supersymmetries for all  $p_1$  and  $p_2$  relevant to the type IIB theory. The integer N denotes the number of common Neumann directions of the (euclidean)  $(p_1 + 1)$ -dimensional and  $(p_2 + 1)$ -dimensional world-volumes. The values marked  $n/a$  require a total of more than eight dimensions transverse to  $x^\pm$  and cannot be described in our light-cone frame. The values marked - do not exist.

#### Type IIB

$p_1$	-1	1			3					5			7
N	0	2	1	0	4	3	2	1	0	6	5	4	8
$p_2 = -1$	16	-	-	0	-	-	-	-	8	-	-	-	-
$p_2 = 1$	0	16	0	8	-	-	0	8	0	-	-	-	-
$p_2 = 3$	8	0	8	0	16	0	8	0	16	-	-	-	-
$p_2 = 5$	0	8	0	16	0	8	0	n/a	n/a	16	0	8	-
$p_2 = 7$	16	0	n/a	n/a	8	n/a	n/a	-	-	0	n/a	n/a	16

The type IIA theories can be described similarly and the following table summarizes the surviving supersymmetries in that case:

#### Type IIA

$p_1$	0		2				4				6	
N	1	0	3	2	1	0	5	4	3	2	7	6
$p_2 = 0$	16	0	-	-	0	8	-	-	-	-	-	-
$p_2 = 2$	0	8	16	0	8	0	-	-	0	8	-	-
$p_2 = 4$	8	0	0	8	0	16	16	0	8	0	-	-
$p_2 = 6$	0	16	8	0	n/a	n/a	0	8	n/a	n/a	16	0

### 4.3.3 Configurations with more than two branes.

When there are three or more  $D$ -branes a smaller fraction of the supersymmetries can be preserved by the solutions.

(a) *4 residual supersymmetries*



A simple example is provided by the configuration of three orthogonal  $D$ -strings with world-sheet orientations in the  $(1, 2); (3, 4); (5, 6)$  directions. The boundary conditions are defined by the matrices,

$$M_{IJ}^1(1) = \text{diag}(-1, -1, 1, 1, 1, 1, 1, 1), \quad M_{ab}^1(1) = \gamma_{ab}^{12} \quad (95)$$

$$M_{IJ}^1(2) = \text{diag}(1, 1, -1, -1, 1, 1, 1, 1), \quad M_{ab}^1(2) = \gamma_{ab}^{34} \quad (96)$$

$$M_{IJ}^1(3) = \text{diag}(1, 1, 1, 1, -1, -1, 1, 1), \quad M_{ab}^1(3) = \gamma_{ab}^{56}. \quad (97)$$

The  $\text{NS} \otimes \text{NS}$  matrices may be transformed by T-duality into those of a  $D$ -instanton and two three-branes with world-volumes that share two directions,  $M_{IJ}^{-1}(1)$ ,  $M_{IJ}^3(2)$  and  $M_{IJ}^3(3)$ . The  $\text{NS} \otimes \text{NS}$  matrices may be transformed by T-duality into those of a  $D$ -instanton and two three-branes with world-volumes that share two directions,  $M_{IJ}^{-1}(1)$ ,  $M_{IJ}^3(2)$  and  $M_{IJ}^3(3)$ . The  $\text{R} \otimes \text{R}$  matrices transform under this T-duality to

$$M_{ab}^{-1}(1) = \delta_{ab}, \quad M_{ab}^3(2) = \gamma_{ab}^{1234}, \quad M_{ab}^3(3) = \gamma_{ab}^{1256}. \quad (98)$$

The common supersymmetries are the ones which have eigenvalues  $+1$ . There are two projectors onto the eigenspaces with eigenvalue  $+1$  for  $M_{ab}^3(2)$  and  $M_{ab}^3(3)$ , respectively,

$$P_{ab}(2) = \frac{1}{2}(1 - \gamma^{1234})_{ab}, \quad P_{ab}(3) = \frac{1}{2}(1 - \gamma^{1256})_{ab}. \quad (99)$$

Since  $[P(2), P(3)] = 0$  the product  $P(2)P(3)$  is also a projector given by

$$(P(2)P(3))_{ab} = \frac{1}{4}(1 - \gamma^{1234} - \gamma^{1256} - \gamma^{3456})_{ab}. \quad (100)$$

Since  $\text{Tr}(P(2)P(3)) = 2$  there are two unbroken undotted supersymmetries. Together with the dotted supersymmetries there is a total of four unbroken supersymmetries – this configuration preserves  $1/8$  of the original 32 supersymmetries.

If this configuration of three 2-instantons is compactified on  $T^6$  in the  $(1, 2, 3, 4, 5, 6)$  directions the result is an instanton in four dimensions which leads to the non-conservation of the three axionic scalar  $\text{R} \otimes \text{R}$  charges associated with  $B_{12}^R, B_{34}^R, B_{56}^R$ .

Another example is a four-dimensional black hole state that can be made by combining three self-dual three-brane world-volumes in the orientations  $(1, 2, 3, 4); (1, 2, 5, 6); (1, 3, 5, 7)$ , interpreting the 1 direction as the time axis and compactifying the directions  $2 - 7$  on  $T^6$ . Since the three-brane does not couple to the dilaton this configuration must correspond to a conventional Reissner–Nordstrom black hole of the type IIB theory. A fourth three-brane in the  $(1, 4, 6, 7)$  direction can also be added. Any of these branes shares its three spatial axes with one of the other ones – this is a stable configuration.

Analogous statements also apply for the Type IIA theory in an obvious manner [36].

## 2 residual supersymmetries

There are also configurations which only preserve 1/16 of the original supersymmetry. One example in the type IIB case is a three-brane with world-volume oriented in the  $(1, 2, 3, 4)$  directions and three mutually orthogonal  $D$ -string world-sheets in the  $(1, 5); (2, 6); (3, 7)$  directions which each intersect the three-brane in one direction. After performing a T-duality transformation that converts the three-brane to a  $D$ -instanton, the projectors for the three one-brane supersymmetries which will be compatible with the unbroken supersymmetries of the three-brane are given by,

$$P(2) = \frac{1}{2}(1 - \gamma^{2345}) \quad (101)$$

$$P(3) = \frac{1}{2}(1 + \gamma^{1346}) \quad (102)$$

$$P(4) = \frac{1}{2}(1 - \gamma^{1247}). \quad (103)$$

These three projectors commute and their product is also a projector,  $P$ , satisfying

$$\text{tr} P = \text{tr}(P(2)P(3)P(4)) = 1, \quad (104)$$

so that there is one undotted unbroken supersymmetry, as well as a similar dotted one. This gives a total of two unbroken supersymmetries. Other similar configurations can easily be constructed.

## 5 Linear and non-linear realizations of space-time supersymmetry

Since a Dirichlet  $p$ -brane breaks translational invariance in the directions transverse to the brane as well as half the space-time supersymmetries [37] there must be a supermultiplet of eight bosonic zero modes (which form a world-volume  $p$ -component vector) and eight fermionic zero modes living in its world-volume. In the case of brane-like solitonic solutions of type IIB supergravity theories these are modes of the bulk fields [38, 39] while in the  $D$ -brane description they are the ground states of the open string.

The unbroken symmetries are realized linearly on the fields whereas the broken symmetries are realized nonlinearly. There is the possibility for some confusion here since in the light-cone frame half the 32 components of the supersymmetry in the type IIB theory are already realized non-linearly. They divide into two groups of 16. One of these groups,  $Q^a$  and  $\tilde{Q}^a$  (associated with the  $SO(8)$  spinor parameters  $\epsilon^a$  and  $\tilde{\epsilon}^a$ ), is realized linearly while the other 16 components,  $Q^{\dot{a}}$  and  $\tilde{Q}^{\dot{a}}$  (associated with the  $SO(8)$  spinor parameters  $\eta^{\dot{a}}$  and  $\tilde{\eta}^{\dot{a}}$ ), is realized non-linearly.

Now we wish to divide these 32 components into groups of 16 in a different manner that accounts for the fact that the boundary state links left-movers and right-movers. The boundary state is annihilated by the combination of supercharges given in (16),

$$Q^{+a} | B \rangle = Q^{+\dot{a}} | B \rangle = 0, \quad (105)$$

which are the unbroken supersymmetries while  $Q^{-b}, Q^{-\dot{b}}$  define the broken supersymmetries. The parameters of the corresponding transformations will be denoted  $\eta_{\pm}^a = \eta^a \pm \tilde{\eta}^a$  and  $\epsilon_{\pm}^{\dot{a}} = \epsilon^{\dot{a}} \pm \tilde{\epsilon}^{\dot{a}}$ .

The algebra of the broken and unbroken supercharges is given by the following commutation relations

$$\begin{aligned} \{Q^{+a}, Q^{-b}\} &= 2\delta^{ab}p^+ & \{Q^{+a}, Q^{-\dot{b}}\} &= \frac{\gamma_{ab}^I}{\sqrt{2}}(p^I + M^{IJ}\tilde{p}^J) \\ \{Q^{-\dot{a}}, Q^{-\dot{b}}\} &= \delta^{\dot{a}\dot{b}}(P^- + \tilde{P}^-) = \delta^{\dot{a}\dot{b}}P_{cl}^-, \end{aligned} \quad (106)$$

with all other anticommutators vanishing.

The open-string light-cone vertex operators are defined only in the case that the momentum carried by the emitted state satisfies  $k^+ = 0$  (our conventions will follow those of [40]). In this case the physical state condition  $\zeta \cdot k$  (where  $\zeta^\mu$  is the physical open-string vector potential) can be satisfied by choosing special transverse polarizations satisfying  $\zeta^I k^I = 0$ , and  $\zeta^- = \zeta^I k^I / k^+$  can be arbitrary. Similarly, the physical state condition on the ground-state open-string spinor field (which has  $SO(8)$  components  $u^a, u^{\dot{a}}$ ) is  $u^a = -2k^I \gamma_{a\dot{a}}^I u^{\dot{a}} / k^+$  which can be satisfied with finite  $u^a$  by choosing the  $\gamma_{a\dot{a}}^I k^I u^{\dot{a}} = 0$ .

In order to avoid awkward singular expressions in the supersymmetry transformations it is convenient to take  $u^{\dot{a}}$  to be proportional to  $k^+$  and define  $v^{\dot{a}} = u^{\dot{a}} / k^+$ , which remains finite as  $k^+ \rightarrow 0$ . The relation between the two  $SO(8)$  components of the ground-state fermion becomes  $u^a = -k^I \gamma_{a\dot{a}}^I v^{\dot{a}}$ . The independent wave functions are then taken to be  $\zeta^I$  and  $v^{\dot{a}}$ .

Supersymmetry transformations on the  $SO(8)$  components of the massless open-string fields take the form,

$$\begin{aligned} \tilde{\zeta}^I &= \eta^a \gamma_{a\dot{a}}^I u^{\dot{a}} \sim 0, & \tilde{\zeta}^I &= \sqrt{\frac{1}{2}} \epsilon^{\dot{a}} \gamma_{a\dot{a}}^I u^a + \frac{\sqrt{2}}{k^+} \epsilon^{\dot{a}} u^{\dot{a}} k^I = k^I \epsilon^{\dot{a}} \gamma_{\dot{a}b}^{IJ} v^{\dot{b}}, \\ \tilde{v}^{\dot{a}} &= \eta^a \gamma_{a\dot{a}}^I \zeta^I, & \tilde{v}^{\dot{a}} &= \frac{1}{k^+} \sqrt{\frac{1}{2}} (\epsilon^{\dot{a}} \gamma_{\dot{a}b}^{IJ} k^I \zeta^J + \epsilon^{\dot{a}} \zeta^I k^I) \end{aligned} \quad (107)$$

(the apparent singularity in the last term does not contribute to the expressions for the on-shell vertex operators).

The massless bosonic open-string vertex operator is given by

$$V_B(\zeta, k) = (\zeta^I B^I - \zeta^-) e^{ikX}, \quad (108)$$

where

$$B^I = \partial X^I - \frac{1}{2} S^a(z) \gamma_{ab}^{IJ} S^b(z) k^J. \quad (109)$$

The massless fermion vertex operator is given

$$V_F(u, k) = (u^a F^a + u^{\dot{a}} F^{\dot{a}}) e^{ikX}, \quad (110)$$

where,

$$F^a = S^a(z), \quad F^{\dot{a}} = \gamma_{a\dot{a}}^I S^a(z) \partial X^I + \frac{1}{6} : \gamma_{a\dot{a}}^I S^a(z) S^b(z) \gamma_{bc}^{IJ} S^c(z) : k^J. \quad (111)$$

The 32 components of the supersymmetry act on the vertex operators in the following way,

$$\begin{aligned} \delta_\eta V_B &= [\eta^a Q^a, V_B(\zeta)] = V_F(\tilde{u}) \\ \delta_\eta V_F &= [\eta^a Q^a, V_F(u)] = V_B(\tilde{\zeta}) \\ \delta_\epsilon V_B &= [\epsilon^{\dot{a}} Q^{\dot{a}}, V_B(\zeta)] = V_F(\tilde{u}) + \epsilon^{\dot{a}} \partial_z W_B^{\dot{a}}(\zeta, k, z) \\ \delta_\epsilon V_F &= [\epsilon^{\dot{a}} Q^{\dot{a}}, V_F(u)] = V_B(\tilde{\zeta}) + \epsilon^{\dot{a}} \partial_z W_F^{\dot{a}}(u, k, z) \end{aligned} \quad (112)$$

The total derivatives  $W_F^{\dot{a}}, W_B^{\dot{a}}$  can give rise to contact terms, which affect the discussion of supersymmetry of the bulk fields but will not enter to the discussion of the open-string sector in this paper. They are given by [40],

$$\begin{aligned} W_B^{\dot{a}} &= \sqrt{2} \gamma_{a\dot{a}}^I \zeta^I S^a e^{ikX} \\ W_F^{\dot{a}} &= \sqrt{2} v^{\dot{a}} e^{ikX} + \frac{\sqrt{2}}{8} (\gamma^{IJ} u)^{\dot{a}} S^b \gamma_{bc}^{IJ} S^c e^{ikX}. \end{aligned} \quad (113)$$

The open-string amplitudes can now be calculated in the cylinder frame where the width of the cylinder in  $\sigma$  is  $\pi$ . The cylinder is chosen to be semi-infinite with the boundary at  $\tau = -\infty$  representing a physical closed-string state  $\langle \Phi |$  that carries the non-zero momentum  $p^+$ . The open-string vertex operators are attached to the boundary at  $\tau = 0$ , and the boundary conditions may be used to express them entirely in terms of left-moving operators. These chiral vertices, which are integrated in  $\sigma$  along are displaced infinitesimally away from the boundary at  $\tau = 0$  with an arbitrary time ordering. All permutations of this ordering are then summed with equal weight [41]. The amplitude with  $n$  bosonic open-string ground states has the form,

$$A_n(\Psi \mid \zeta_1, \dots, \zeta_n) = \int d\sigma_1 \dots d\sigma_n \langle \Psi \mid V_B(\zeta_1, k_1, z_1) \dots V_B(\zeta_n, k_n, z_n) \mid B \rangle. \quad (114)$$

A supersymmetry transformation of this amplitude is obtained by substituting a transformed wavefunction  $\tilde{\zeta}_1$  or  $\tilde{\zeta}_1$  into  $A_n$ . Using (112) the transformed vertex operator

can be written as a commutator of a supersymmetry generator and a fermionic vertex operator. Thus, for the linearly realized components of the conserved supersymmetry the vertex,

$$V_B(\tilde{\zeta}) = \eta_+^a Q^{+a} V_F(u) - V_F(u) \eta_+^a Q^{+a}, \quad (115)$$

can be inserted into (114) and the  $Q^+$  in the first term acts to the left on the closed string state  $\Psi$  giving a transformed state,  $\delta_{\eta^\pm} \Psi$ . The  $Q^+$  in the second term is moved to the right and gives transformed open string vertex operators until it hits the boundary where it is annihilated by the boundary state. Thus the conserved supersymmetry relates S-matrix elements with  $n + 1$  bosonic states (including the one (bosonic) closed-string end-state) to elements with  $n - 1$  bosonic and 2 fermionic states,

$$\begin{aligned} A_n(\Psi | \tilde{\zeta}_1, \zeta_2, \dots, \zeta_n) &= A_n(\delta_{\eta^+} \Psi | u_1, \zeta_2, \dots, \zeta_n) + A_n(\Psi | u_1, \tilde{u}_2, \zeta_3 \dots, \zeta_n) \\ &+ \dots + A_n(\Psi | u_1, \zeta_2, \dots, \zeta_{n-1}, \tilde{u}_n). \end{aligned}$$

This corresponds to the linearly realized supersymmetry which is not broken by the boundary state. A similar analysis applies to the non-linearly realized conserved supercharge,  $Q^{+\dot{a}}$  with  $\tilde{\zeta}$  and  $\tilde{u}$  replaced by  $\tilde{\tilde{\zeta}}$  and  $\tilde{\tilde{u}}$ .

The supercharge  $Q^{-a}$  is not annihilated by the boundary so that similar manipulations for these supercharges leave a residual term. This is proportional to  $\eta_-^a Q^{-a} |B\rangle$ , which has the form of the fermion emission vertex (110) acting on the boundary, in which the supersymmetry parameter  $\eta_-^a$  is the wave function. Therefore, the amplitude with  $n + 1$  bosonic states is related by the  $Q^-$  supersymmetry to a sum of terms with  $n - 1$  bosons and two fermions, together with an extra term which has an extra zero-momentum fermion insertion – it has a total of  $n$  bosons and two fermions,

$$\begin{aligned} A_n(\Psi | \tilde{\zeta}_1, \zeta_2, \dots, \zeta_n) &= A_n(\delta_{\eta^-} \Psi | u_1, \zeta_2, \dots, \zeta_n) + A_n(\Psi | u_1, \tilde{u}_2, \zeta_3 \dots, \zeta_n) \\ &+ \dots + A_n(\Psi | u_1, \zeta_2, \dots, \zeta_{n-1}, \tilde{u}_n) + A_{n+1}(\Psi | u_1, \zeta_2, \dots, \zeta_n, \eta^-) \end{aligned} \quad (116)$$

This is the S-matrix statement of the nonlinear realization of the spontaneously broken  $Q^-$  supersymmetry [41].

The corresponding analysis with the non-linearly realized supercharge  $Q^{-\dot{a}}$  leads to the same relationship between amplitudes but with  $\tilde{\zeta}$  and  $\tilde{u}$  replaced with  $\tilde{\tilde{\zeta}}$  and  $\tilde{\tilde{u}}$ . Higher-order terms give rise to S-matrix elements with arbitrary numbers of soft fermions.

Note that (17) and (33) implies  $Q^{-a} = \hat{\theta}^a$ , hence the nonlinearly realized supersymmetry charge  $Q^{-a}$  acts as the modified Grassmann parameter  $\hat{\theta}^a$  on the boundary state  $|B, +\rangle$  which is the bottom component of a superfield expanded in  $\hat{\theta}^a$ . The vertex operator for the emission of zero momentum Goldstinos is given by  $\eta_-^a \theta^a |B_0, \theta, +\rangle$ , which indicates that the nonlinearly realized supersymmetry acts as a shift on the Goldstino field.

One important subtlety in this analysis is that contact terms have to be carefully accounted for. These arise from the derivative terms in the algebra (112) but in the open-string amplitudes considered here they integrate to zero. However, they are an important feature of more general amplitudes with closed-string vertex operators.

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